ON THE DIFFERENTIAL GEOMETRY OF RULED SURFACES IN 4-SPACE AND CYCLIC SURFACES IN 3-SPACE*

BY

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Introduction

This paper is devoted to a further development and application of some of the ideas contained in my two earlier papers *On the projective differential geometry of n-dimensional spreads generated by* ∞^1 *flats*† and *On the projective differential classification of n-dimensional spreads generated by* ∞^1 *flats*,‡ which will be referred to as P. D. G. and P. D. C., respectively.

In P. D. C., §§ 42–59, a study was made of ruled surfaces (2-spreads) in 4-space and their correlative loci, planar 3-spreads. The results there obtained are now supplemented and made more significant by a discussion, in Part I of the present paper, of the relation between these loci and their transversal loci.

The combined and to a certain extent completed theory so developed then finds an application, in Part II, to the differential geometry of cyclic (circled) surfaces and pairs of curves in space of three dimensions. This application depends on the known correspondence between the projective point-geometry of 4-space and the descriptive sphere-geometry of 3-space, by virtue of which a straight line of 4-space represents a circle of 3-space and a plane of 4-space represents a pair of points of 3-space.

Although cyclic surfaces have been much studied before from other points of view, the present method throws a considerable additional light on their structure, as the new theorems will testify. Among the striking results is the fact that there exists on every cyclic surface that is neither annular nor anallagmatic a certain unique and fundamental pair of (quasi-asymptotic) curves. See § 26.

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† Annali di Matematica Pura ed Applicata, ser. 3, vol. 19 (1912), pp. 205–249.

‡ American Journal of Mathematics, vol. 37 (April, 1915).

PART I. TRANSVERSAL RULED SURFACES AND PLANAR 3-SPREADS IN 4-SPACE. TRANSVERSAL RULED SURFACES

1. Let S be a skew ruled surface immersed in 4-space, and let the homogeneous coördinates of its points be expressed in terms of the parameters u and v by means of the equations

$$X_j = A_j(u) + v B_j(u) \qquad (j = 1, \cdots, 5), \tag{1}$$

where $A_j(u)$ and $B_j(u)$ are continuous functions possessing continuous first, second, and third derivatives. Symbolically, we write $S = [A, B]$, and

$$X = A(u) + v B(u). \tag{2}$$

The functions A_j and B_j satisfy a linear homogeneous differential equation of the second order, which we assume to be in the canonical form*

$$\frac{d^2 A}{du^2} + \frac{d}{du}(l B) + mB = 0$$

or, more briefly,

$$A'' + (l B)' + mB = 0, \tag{3}$$

where l and m are functions of u. They also satisfy a differential equation of the third order, but this one of the second order is the only one that we are concerned with at present. The results hold equally well whether we are operating in the real or the complex domain.

2. It is known that in 4-space there exists one and only one line meeting three given skew lines; this we call their *transversal.* Consider three neighboring lines of the surface S, one of which, a, is fixed and non-singular,† and the other two approach the first in any manner along the surface. Their transversal will approach a limiting position c, which we shall speak of as meeting three consecutive lines of S. The locus of c, as a describes S, is a new ruled surface $\bar{S}$, the *transversal* of S.‡ The surface $\bar{S}$ is the only ruled surface whose generators have three-point contact with S.

3. We proceed to set up the actual equations of $\bar{S}$; we shall then discover an interesting relation between S and $\bar{S}$.

Let P be the point of intersection of a and c, and π their common plane; π is obviously tangent to S at P. Let a_1, $a_2\,(= a)$, and a_3 be three con-

* See P. D. C., § 42, equation (17′).

† A generator is singular if it either meets its consecutive generator or lies in the 3-flat determined by its two consecutive generators.

‡ Cf. H. Mohrmann, *Ueber die windschiefen Linienflächen im Raume von vier Dimensionen und ihre Haupttangentenflächen als reciproke Linienflächen,* Archiv der Mathematik und Physik, ser. 3, vol. 18 (1911), pp. 66–68; E. Bompiani, *Alcune proprietà proiettivo-differenziali dei sistemi di rette negli iperspazi,* Rendiconti del Circolo Matematico di Palermo, vol. 37 (1914), p. 4.

secutive generators of S, and let a_1, a_2 determine the 3-flat f_{12} and a_2, a_3 the 3-flat f_{23}; a and c will both lie in f_{12}, and also in f_{23}; hence they will lie in the plane of intersection of f_{12} and f_{23}, and the latter must be π. But f_{12} and f_{23} are generators of $S^{1,0}$;* and it follows that π is a generator of the planar developable $S^{1,1}$* and that P is the corresponding point of the curve T.†

Let d be the *characteristic* of the plane π, i. e., the line of intersection of π and its consecutive plane; d is then a generator of $S^{1,2}$. Let b be the tangent to T at the point P, i. e., the corresponding generator of T^1. The lines b and d both pass through P and lie in π. Hence a, b, c, and d belong to a pencil of lines. We wish to prove that *a and c are harmonic conjugates with respect to b and d.*

4. **Proof.** In order to indicate the fact that a is the line joining the points $A_j(u)$ and $B_j(u)$ for a particular value of u, we write

$$a = (A, B). \tag{4}$$

Similarly, $f_{12} = (A, B, A', B')$ and $\pi = (A, B, A')$. In view of (3), it is clear that

$$d = (A, A' + lB). \tag{5}$$

Since P is the point of intersection of d and a, $P = (A)$. Hence

$$b = (A, A'). \tag{6}$$

Since the transversal c belongs to the pencil determined by a and b, it meets the line joining the points (A') and (B) in some point $(A' + kB)$; that is, $c = (A, A' + kB)$, where k is to be determined so that c will have three-point contact with the surface S, and therefore with some curve K on S. The curve K will pass through the point P, for which $v = 0$, and will be given by the equation (2), if we assume u and v in this equation to be functions of a third variable t.

Now three consecutive points of K are determined by (X), $(X_u\, du + X_v\, dv)$, and $(X_{uu}\, du^2 + 2X_{uv}\, dudv + X_{vv}\, dv^2 + X_u\, d^2 u + X_v\, d^2 v)$, where $X_u = \partial X/\partial u$, etc. By reason of equation (3), these three points become, for $v = 0$, (A), $(A'\, du + Bdv)$, and $(A''\, du^2 + 2B'\, dudv + A'\, d^2 u + Bd^2 v)$. They are to be collinear and are to lie on the line c. The first two will lie on c, if $dv/du = k$.

In order that the third point may lie on c, it is sufficient to determine $dv/du = k$ so that $A''\, du^2 + 2B'\, dudv = (A'' + 2kB')\, du^2$ will lie in the tangent plane $\pi = (A, B, A')$; for since $A'\, d^2 u + Bd^2 v$ lies in π, $d^2 v/d^2 u$ can then be determined so that the third point will be collinear with the

* For the exact definitions of $S^{1,0}$, $S^{1,1}$, T, etc., see P. D. G., § 29, and P. D. C., §§ 42–44.

† Bompiani, loc. cit., pp. 1, 4, 5, calls T a quasi-asymptotic curve, and denotes it by $\gamma_{2,2}$.

other two.* In other words, $A'' + 2kB'$ must be equal to a linear homogeneous function of A, B, and A'. But the only existing relation of this kind is (3); hence $k = \frac{1}{2} l$, and

$$c = (A, A' + \tfrac{1}{2} lB). \tag{7}$$

Comparing (4), (5), (6), and (7), we see that the cross-ratio of the lines a, b, c, d is the same as that of the points A', B, $A' + lB$, and $A' + \frac{1}{2} lB$, which is equal to 2. Hence the lines are harmonic.

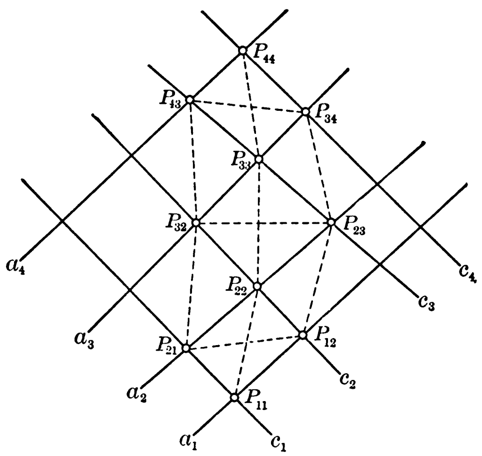

FIG. 1

5. We have now proved the

THEOREM.† *Every non-singular generator of a skew ruled surface S in 4-space and the corresponding generator of its transversal surface $\bar{S}$ are harmonic conjugates with respect to two lines b and d, of which b is the corresponding tangent to the curve of contact of S and $\bar{S}$, and d is the characteristic of their common tangent plane.*

Incidentally, (7) shows that the equations of $\bar{S}$ are

$$X_j = A_j(u) + v[A_j'(u) + \tfrac{1}{2} l(u) B_j(u)] \quad (j = 1, \cdots, 5).$$

6. It may not be amiss to add another proof, or rather quasi-proof, of this theorem, which, although not rigorous, has the advantage of accentuating its geometric significance.

In Fig. 1 let a_1, a_2, a_3, a_4 be four neighboring generators of S; let c_2 be

* Cf. C. L. E. Moore, *Surfaces in hyperspace which have a tangent line with three-point contact passing through each point*, Bulletin of the American Mathematical Society, vol. 18 (1912), p. 284.

† This theorem has also been found by Professor Bompiani. In answer to a recent letter in which I mentioned the theorem to him, and asked him whether he had noticed it, he replied in the affirmative.

the transversal of a_1, a_2, and a_3, meeting them in the points P_{12}, P_{22}, and P_{32}, respectively; and more generally, let c_i be the transversal of a_{i-1}, a_i, and a_{i+1}, meeting them in the points $P_{i-1,\,i}$, P_{ii}, and $P_{i+1,\,i}$. In the figure actual points of intersection are indicated by circles; the others are only apparent. Then c_1, c_2, c_3, c_4 are lines which may be regarded as becoming, by a limit process, generators of $\bar{S}$. Similarly P_{11}, P_{22}, P_{33}, P_{44} may be regarded as becoming points of the curve of contact T; or, if we wish, T may be generated by P_{12}, P_{23}, P_{34}, etc.; $P_{12}\,P_{23}$ then represents one of the tangents to T. Let π_i be the plane determined by a_i and c_i, that is, the common tangent plane to S and $\bar{S}$ at the point P_{ii}; then π_2 is the plane of the quadrilateral $P_{12}\,P_{21}\,P_{32}\,P_{23}$ and π_3 the plane of the quadrilateral $P_{23}\,P_{32}\,P_{43}\,P_{34}$. Hence π_2 and π_3 meet in the line $P_{23}\,P_{32}$, which therefore represents one of the generators of $S^{1,2}$.

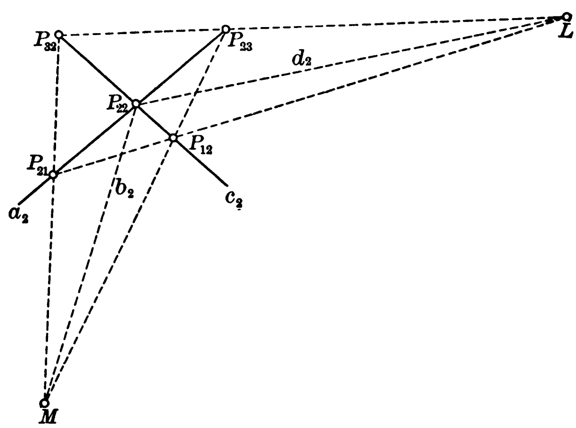

FIG. 2

In Fig. 2 we have the complete quadrangle determined by $P_{12}\,P_{21}\,P_{32}\,P_{23}$. It is evident that the tangent b_2 to T at the point P_{22} can be represented by $P_{22}\,M$ instead of $P_{12}\,P_{23}$, and that the corresponding generator d_2 of $S^{1,2}$ can be represented by $P_{22}\,L$ instead of $P_{23}\,P_{32}$. It follows that a_2, b_2, c_2, d_2 form a harmonic pencil.

7. Let us now examine the significance of this theorem for the three classes (a), (b), (c) of skew ruled surfaces, as they were defined in P. D. C., §§ 45–51.

If S belongs to the most general class (a), T is distinct from the edge of regression $S^{1,3}$ of $S^{1,2}$, and the lines b and d are distinct; hence *the transversal surface* $\bar{S}$ *is another skew ruled surface of the class* (a) and the same subclass (a_{ij}) as S itself; S is then the transversal surface of $\bar{S}$, so that the relation between S and $\bar{S}$ is a reciprocal one. All the skew surfaces therefore whose generators lie in the tangent planes to a developable $S^{1,2}$ and pass through the corresponding points of a curve T lying on $S^{1,2}$ (T not coinciding with

$S^{1,3}$), arrange themselves in pairs of transversals. Every surface of this set touches every other surface of the set along the curve T; while in the case of two transversal surfaces of the set every generator of one has three-point contact with the other.

If S belongs to class (b), T coincides with $S^{1,3}$, and b coincides with d. Hence c also coincides with d, and $\bar{S}$ coincides with $S^{1,2}$; that is, *the transversal surface of a skew ruled surface of class* (b) *is a non-conical developable.* All the surfaces, ∞^{f_1} in number, whose generators pass through the points of $S^{1,3}$ and lie in the corresponding tangent planes of $S^{1,2}$, have the same transversal surface $S^{1,2}$. Reciprocally, it is easy to see that all these surfaces, and these alone, are transversal surfaces of $S^{1,2}$. That is, a non-conical developable surface $S^{1,2}$ immersed in F_4 has an infinite number of transversal surfaces, which, apart from $S^{1,2}$ itself, are all skew surfaces of class (b).

If S belongs to class (c), its generators all meet a fixed line d, with which b and c coincide. So the transversal surface $\bar{S} = S^{1,2}$ is of range 0 and reduces to the line d itself.

8. One or two special cases may be mentioned in passing. If T is an orthogonal trajectory of the generators of $S^{1,2}$, the corresponding generators of S and $\bar{S}$ make equal angles with the corresponding tangents to their curve of contact T. If, on the other hand, the corresponding lines of S and $\bar{S}$ are perpendicular to one another, they bisect the interior and exterior angles between the corresponding tangents to T and the generators of $S^{1,2}$.

Moreover, if a and c are constantly parallel, the curve T lies entirely at infinity, the point P and the line b are at infinity, and the line d is parallel to a and c and lies half-way between them. S and $\bar{S}$ then belong to class (a) and subclass (a_{12}) or (a_{22}). Conversely, if $S^{1,2}$ is any developable surface, either non-conical or a point-cone, and if every line a of S is parallel to the corresponding line d of $S^{1,2}$, then every line c of the transversal surface $\bar{S}$ is also parallel to d, and a and c lie on opposite sides of d (in the plane π) and are equidistant from d.

9. In order to illustrate the case in which two transversal surfaces are of class (a), we choose for S the rational quartic surface of subclass (a_{22}) described in P. D. C., § 48, whose equations are

$$x_1 : x_2 : x_3 : x_4 : x_5 = v : 1 : u + v : u^2 + 2uv : u^3 + 3u^2 v. \tag{8}$$

Eliminating u and v, we get the equations

$$x_2 x_4 = x_3^2 - x_1^2, \qquad x_2^2 x_5 = (x_3 - x_1)^2 (x_3 + 2x_1), \tag{9}$$

which represent not only S, but also the plane $x_2 = x_3 - x_1 = 0$, counted twice. The differential equation (3), which the directrices A and B satisfy, is in this case $A'' - B' = 0$. Hence $\bar{S} = [A, A' - \frac{1}{2}B]$, and it follows that the equations of $\bar{S}$ are obtainable from (8) or (9) by merely replacing x_1 by $-x_1$.

TRANSVERSAL PLANAR 3-SPREADS IN F_4

10. By the use of duality we can now easily develop the corresponding theory of skew planar 3-spreads in F_4 and their transversal 3-spreads. The metrical applications are somewhat more interesting than they were for the ruled surfaces. Let S be any such 3-spread, not a cone. Its simpler properties and classification are given in P. D. C., §§ 52–59. Two consecutive generating planes of S meet in a point, and these points generate a curve $S_{1,0}$.

Now any three planes in F_4 which meet each other by pairs in points determine one and only one plane meeting each of them in a line, namely the plane connecting the three points in which they meet; this plane we call their *transversal*. The transversal of three consecutive planes of S will generate a new planar 3-spread $\bar{S}$, the *transversal* 3-*spread* of S.

Let α be a non-singular plane of S and γ the corresponding plane of $\bar{S}$. They will meet in a line a and determine a 3-flat f; a is then a tangent to the curve $S_{1,0}$, that is, a generator of $S_{1,1}$, while f is the corresponding tangent 3-flat to S (at every point of a), and therefore generates the spread T.* The plane β, in which f meets the consecutive generator of T, is a generator of $T_{1,0}$. Let δ be the corresponding osculating plane of $S_{1,0}$, or generator of $S_{1,2}$. Then β and δ both lie in the 3-flat f and pass through the line a. Hence the four planes α, β, γ, δ form a pencil, which by the correlative of the theorem of § 5 we see to be harmonic.

11. Evidently S and $\bar{S}$ touch each other at every point of the developable surface $S_{1,1}$; and f is their common tangent 3-flat at every point of the generator a of $S_{1,1}$; that is, every line lying in f and meeting a has 2-point contact with both 3-spreads. Moreover, every line in the plane γ has 3-point contact with S and every line in the plane α has 3-point contact with $\bar{S}$. Summarizing, we have the

THEOREM. *Every non-singular generator of a non-conical skew planar 3-spread S in 4-space and the corresponding generator of its transversal 3-spread $\bar{S}$ are harmonic conjugates with respect to two planes β and δ, of which δ is the corresponding tangent plane to the developable surface along which S and $\bar{S}$ touch each other and β is the characteristic of their common tangent 3-flat.*

When the equations of S are given in the canonical form† $S = [A, A', B]$, where A and B satisfy a differential equation of the type

$$A''' + (lB)' + mB + nA = 0,$$

then it is easy to see that $\bar{S} = [A, A', A'' + \frac{1}{2}lB]$.

* See P. D. C., § 54.

† See P. D. C., § 52.

12. If S belongs to the class (a), the planes α, β, γ, δ are distinct, and $\bar{S}$ is a skew 3-spread belonging to the same class (a) and the same subclass (a_{ij}) as S. Then S is the unique transversal 3-spread of $\bar{S}$.

If S belongs to the class (b), the planes β, γ, δ coincide, and $\bar{S}$ is developable, coinciding with $S_{1,2}$. Conversely, every non-conical developable 3-spread $S_{1,2}$ immersed in F_4 has an infinite number of transversal 3-spreads, which, apart from $S_{1,2}$ itself, are all skew 3-spreads of class (b).

If S belongs to the class (c), its planes all meet a fixed plane δ in lines; β and γ coincide with δ, and $\bar{S}$ degenerates into the plane δ itself.

A fixed 3-flat f', not containing any of the lines of contact a, will intersect S and $\bar{S}$ in two ordinary ruled surfaces R and $\bar{R}$, such that every line of R has 3-point contact with $\bar{R}$ and vice versa; f' will intersect $S_{1,1}$ in the curve of contact of R and $\bar{R}$. The four harmonic planes α, β, γ, δ will be met by f' in four harmonic lines a, b, c, d. Since a and c give the two asymptotic directions at a point of R (and of $\bar{R}$), b and d give a pair of conjugate directions.

13. We now proceed to impose certain metrical restrictions on S and $\bar{S}$. If the planes β and δ are perpendicular, they bisect the angles between α and γ; while if α and γ are perpendicular, they bisect the angles between β and δ.

On the other hand if α and γ are constantly parallel, the curve $S_{1,0}$ lies entirely at infinity, the line a and the plane δ lie at infinity, and the plane β is parallel to α and γ, lying halfway between them. Then S and $\bar{S}$ belong to subclass (a_{21}) or (a_{22}). Every plane of S (and of $\bar{S}$) is half-parallel* to its consecutive plane, but not, in general, to its other planes. Conversely, if consecutive planes of S are half-parallel, while non-consecutive planes are not, then every plane of S is completely parallel to the corresponding plane of its transversal 3-spread $\bar{S}$.

Suppose now that any two planes of a planar 3-spread S, whether consecutive or not, are half-parallel. Then the planes of S meet the 3-flat at infinity in a system of lines, every two of which intersect each other; hence they either pass through a fixed point or lie in a fixed plane. In the first case S is a conical (or rather cylindrical) skew planar 3-spread; its planes are half-parallel at the same infinite point. In the second case the planes of S are half-parallel at different infinite points; the curve $S_{1,0}$ generated by the infinite points at which consecutive planes of S are half-parallel is a plane curve. Hence S is a non-conical skew planar 3-spread of class (c).

14. Finally, letting f' (see § 12) be the 3-flat at infinity, we shall consider the case in which R and $\bar{R}$ have certain special properties, while $S_{1,0}$ is a finite curve.

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* Half-parallel planes in F_4 meet in an infinite point, while completely parallel planes meet in an infinite line. See Schoute, *Mehrdimensionale Geometrie*, Erster Teil (1902), pp. 20–40.

Now the metrical geometry of f' may be regarded, for our purpose, as a riemannian or elliptic geometry. Hence the interesting case presents itself in which the generators of R are *paratactic* lines* (clifford parallels), that is, they belong to a right (or left) paratactic congruence. We shall apply the term paratactic also to the planes of S. In other words, *two planes lying in a 4-flat and meeting in a finite point are paratactic, if all the lines of one are equally inclined to the other.*† Obviously, two planes right paratactic to a third plane are right paratactic to each other, whether they have the same points of intersection or not. Hence if *consecutive* planes of S are paratactic, *all* its planes are paratactic. Then S must be a skew 3-spread, but may be a point-cone or a non-conical spread of class (a), (b), or (c).

If it is of class (c), the line d, as defined in § 12, is a fixed line, with which b and c coincide, and the lines a form one regulus (system of generators) R of a clifford surface. If S belongs to the class (a), and to any one of the four subclasses, it may happen that the planes of the transversal 3-spread $\bar{S}$ are left paratactic at the same time that those of S are right paratactic, or vice versa. If so, it is clear that *their loci at infinity, R and $\bar{R}$, are the two reguli of one and the same clifford surface.*

Part II. The descriptive differential geometry of cyclic surfaces and pairs of curves in 3-space. Cyclic surfaces

15. By a cyclic (or circled) surface is meant a surface generated by circles. Since a circle may be regarded as the envelope of a pencil of spheres, it is natural to regard the geometry of cyclic surfaces as included in sphere-geometry. The theorems of sphere-geometry are of three essentially different kinds.

First there are the ordinary metrical theorems, which are invariant under the 6-parameter group of movements, which are point-transformations carrying every sphere into a congruent sphere. Then there are the more fundamental properties invariant under the 10-parameter group of conformal transformations, which are also point-transformations carrying spheres into spheres, but not necessarily into congruent spheres. Finally, there are the most fundamental properties of all, the descriptive properties, which are invariant under the 24-parameter group of sphere-transformations; these are not point-transformations at all, but carry point-spheres, in general, into proper spheres.

Our purpose is to study the differential geometry of cyclic surfaces from this last point of view. The previous writers on the subject have confined themselves almost exclusively to metrical or conformal properties. In par-

* See Coolidge, *Non-Euclidean Geometry* (1909), p. 99.

† See Schoute, *Mehrdimensionale Geometrie*, Erster Teil (1902), p. 72.

ticular, Enneper's classification* of cyclic surfaces, as developed by Demartres, Cosserat and Moore,* is invariant under the conformal group, but not under the descriptive group. The purely descriptive classification which we shall find is entirely different from Enneper's except in the broad distinction between annular and non-annular surfaces. We shall find eleven classes of surfaces, of which four are annular and seven non-annular.

16. It will be unnecessary for our purpose to use pentaspherical coördinates. It will be sufficient to write the equation of a sphere A in the form

$$A_1(x^2+y^2+z^2)+A_2x+A_3y+A_4z+A_5=0, \tag{10}$$

and to regard $A_1:A_2:A_3:A_4:A_5$ as its homogeneous coördinates. We shall use the word sphere in a broad sense, to include the plane and the point (point-sphere). The non-singular linear sphere-transformations,

$$kA_i'=\sum_{j=1}^{5}\alpha_{ij}A_j \qquad (i=1,\cdots,5), \tag{11}$$

form the descriptive group under which we are to operate. But the coördinates A_i may be interpreted as representing a point in 4-space F_4, and (11) as a collineation in F_4. Hence the descriptive sphere-geometry of 3-space corresponds to the projective point-geometry of 4-space.

It may be remarked, in passing, that the conformal group in F_3 corresponds to the group of collineations in F_4 that leave a certain quadric 3-spread invariant, and therefore that the conformal geometry of F_3 corresponds to the metrical geometry of a non-euclidean F_4.

Any two spheres A and B determine a pencil of spheres enveloping a circle a, just as two points in F_4 determine a range of points lying on a line. If the coördinates A_i and B_i of these two spheres are functions of a parameter u, the circle a will generate a cyclic surface† S, corresponding to a ruled surface in F_4. Hence the theorems derived in §§ 1–9 of this paper and P. D. C., §§ 42–51, on the projective differential geometry of ruled surfaces in F_4, can be translated immediately into theorems on the descriptive differential geometry of cyclic surfaces in F_3.

17. In the correspondence between these two geometries a point of F_4

* A. Enneper, *Die cyklischen Flächen*, Zeitschrift für Mathematik und Physik, vol. 14 (1869), pp. 393–421. G. Demartres, *Sur les surfaces à génératrice circulaire*, Annales de l'école normale supérieure, ser. 3, vol. 2 (1885), p. 142. E. Cosserat, *Sur le cercle considéré comme élément générateur de l'espace*, Annales de Toulouse, vol. 3 (1889), E, pp. 1–12, 25–27. C. L. E. Moore, *Lines in space of four dimensions and their interpretation in the geometry of the circle in space of three dimensions*, American Journal of Mathematics, vol. 33 (1911), pp. 143–147. See also Demartres, *Cours de géométrie infinitésimale* (1913), pp. 308–311, and R. V. Lilienthal, *Vorlesungen über Differentialgeometrie*, zweiter Band, erster Teil (1913), pp. 102–115.

† Naturally circles here include straight lines, and cyclic surfaces include ruled surfaces.

corresponds to a sphere of F_3, a line of F_4 to a circle (pencil of spheres) of F_3, a plane of F_4 to a pair of points of F_3; the points of a plane correspond to the spheres of a linear congruence, or net, which are the spheres orthogonal to a given circle, or passing through a given pair of points, the foci of the given circle; the lines of the plane correspond to the circles passing through the given pair of points, or in bi-involution* with the given circle. A pencil of lines corresponds to a pencil of circles (lying on a sphere and passing through two points of the sphere). The points of a 3-flat correspond to the spheres of a linear complex, which are the spheres orthogonal to a given sphere; the planes of the 3-flat correspond to the pairs of points inverse with respect to the given sphere. A pencil of planes (passing through a line and lying in a 3-flat) corresponds to a *pencil of point-pairs*, by which is meant a system of point-pairs on a circle, which are inverse with respect to a given sphere orthogonal to the circle; the lines joining the pairs of points therefore pass through the center of the given sphere.

Orthogonality, to be sure, is not a descriptive property of spheres and circles, and so, properly speaking, has no place in this discussion. Nevertheless, we have introduced it for convenience in the statement of certain results. It will be readily seen that orthogonality is not essential to the argument that is to follow, and that by circumlocution all mention of it could be avoided.

On the basis of the known classification of ruled surfaces in F_4, we proceed to find the corresponding classification of cyclic surfaces in F_3. Let Σ and S be corresponding ruled and cyclic surfaces, respectively. The surface Σ will be a developable or a skew surface, according as every two consecutive lines meet in a point or not. Hence S will be an annular or a non-annular surface, according as every two consecutive circles are cospherical or not.

Annular surfaces

18. Since a developable Σ is either a point-cone or is generated by the tangents to a curve Σ_1, therefore an annular surface S is either a sphere or the envelope of a family S_1 of ∞^1 spheres.

In the latter case Σ_1 is either a plane curve, a 3-space curve or a 4-space curve; that is, Σ belongs to one of the types† $(01\bar{1}0)$, $(01\bar{1}10)$, or $(01\bar{1}110)$. In F_3 this means that any two consecutive circles, a and b, of S determine a net of spheres containing the pencils of spheres passing through a and b, and that if this net is fixed, S is of type $(01\bar{1}0)$. If this net is variable, then three consecutive circles of S will similarly determine a linear complex of spheres, which, if fixed, gives the type $(01\bar{1}10)$, and if variable, the type $(01\bar{1}110)$.

* See Koenigs, *Contributions à la théorie du cercle dans l'espace*, Annales de Toulouse, vol. 2 (1888), F, p. 9.

† See P. D. C., §§ 33–35.

Accordingly, annular surfaces enveloped by ∞^1 spheres are of three distinct types, as follows.

19. **Type** $(01\bar{1}0)$. Since Σ^1 is a fixed plane, S^1 is a fixed pair of points, and S may be described as a surface generated by circles passing through two fixed points, or in bi-involution with a fixed circle c. If one of the fixed points is at infinity, the generators of S become straight lines and S becomes a cone. If the fixed circle c is a straight line, S becomes a surface of revolution; whereas, if c is not a straight line, S may be described as an *anallagmatic surface whose deferent is a plane curve.* If the deferent is a conic, the surface becomes a cyclide of Dupin.

20. **Type** $(01\bar{1}10)$. Here S^1 consists of a pair of curves enveloped by the circles of S. Hence S is generated by a circle constantly tangent to a pair of curves. One of these curves may degenerate into a fixed point. Moreover, since Σ^2 is a fixed 3-flat, the circles of S are all orthogonal to a fixed sphere σ, and the point-pairs of S^1 are inverse with respect to σ. If σ is not a plane, S may be described as an *anallagmatic surface whose deferent is a space curve.* The simplest algebraic example is the one in which the deferent is a twisted cubic. If one of the component curves of S^1 degenerates into the point at infinity, S becomes a developable surface.

21. **Type** $(01\bar{1}110)$. The circles of S are again tangent to a pair of curves S^1, as in the preceding case, but are now no longer orthogonal to a sphere; so S cannot be anallagmatic. Neither component curve of S^1 can degenerate into a point; S cannot be a ruled surface. Every three consecutive circles of S determine one sphere to which they are orthogonal, and these spheres envelope another annular surface T, also of type $(01\bar{1}110)$, correlative to S. Every circle of T is in bi-involution with two consecutive circles of S, and conversely. The foci of every circle of T are therefore the corresponding pair of points of S^1.

It may happen that every point-pair of S^1 is a pair of coincident points, and that the generators of S are the osculating circles of a single curve S^1. But this special case, while listed as a distinct type in Enneper's classification, is not a distinct type in our descriptive classification; for a pair of coincident points will not be carried into a pair of coincident points by the general sphere-transformation.

Non-annular cyclic surfaces

22. In this case two consecutive circles of S are not, in general, cospherical, and therefore cannot meet in a pair of points. They may meet in a single point, to be sure, but that is not a descriptive property, and so does not determine a distinct class of surfaces from our standpoint.

In F_4 the corresponding ruled surface Σ is skew, and so has ∞^2 tangent planes, as well as ∞^2 points. Every plane containing a given generator is tangent to the surface at some point of the generator; and these planes are projectively related to their points of contact. Hence S is enveloped in a very special manner* by the system of ∞^2 spheres passing through its generating circles; this system therefore consists of ∞^1 pencils. Every sphere of the system is a bitangent to the surface, touching it at a pair of points on the generator through which it passes. The pairs of points of contact of the spheres of a pencil containing a given circle of S form a pencil of point-pairs on the circle, projective to the pencil of spheres.

Just as two consecutive lines of Σ determine a 3-flat f, so two consecutive circles of S determine a linear complex of spheres containing the pencils of spheres passing through them, and therefore also a single sphere σ to which they are orthogonal. The point-pairs of the pencil just mentioned are inverse with respect to σ. If f is fixed, Σ is a 3-space surface, of type $(0\bar{2}0;)$† while if f is variable, Σ is a 4-space surface, of type $(0\bar{2}10)$.† This gives us the two principal classes of non-annular cyclic surfaces S. They are of type $(0\bar{2}0)$ or of type $(0\bar{2}10)$, according as their generators are orthogonal to a fixed sphere or not.

23. **Type** $(0\bar{2}0)$. The surface S may be described, in general, as an *anallagmatic surface, whose deferent is a skew ruled surface.* An exceptional case is that in which S has a plane of symmetry, to which its generators are orthogonal. Type $(0\bar{2}0)$ includes the most general class of cyclic, anallagmatic surfaces. The simplest algebraic example is that in which the deferent is a quadric surface and S is a cyclide; Σ is then also a quadric surface. If the fixed orthogonal sphere σ is a point-sphere, S is generated by circles passing through a fixed point, and in particular, if σ is at infinity, S degenerates into a skew ruled surface.

Since three consecutive lines of Σ determine an osculating quadric surface $\bar{\Sigma}$, and have an infinite number of common transversals, which are the generators of the other system of $\bar{\Sigma}$, therefore three consecutive circles a, b, c, of S determine an osculating cyclide $\bar{S}$ and have an infinite number of *transversal circles*, namely circles cospherical with a, with b, and with c; these transversals are the generators of the other system of the cyclide $\bar{S}$.

Type $(0\bar{2}10)$

24. We now come to the most general type of cyclic surfaces S, those which are neither annular nor anallagmatic. Two consecutive circles of S are not, in general, cospherical, and three consecutive circles are not, in general,

* See Demartres, loc. cit., pp. 147–148.

† See P. D. C., §§ 36 and 39.

orthogonal to the same sphere. Let

$$A_1(x^2 + y^2 + z^2) + A_2 x + A_3 y + A_4 z + A_5 = 0 \tag{10}$$

and

$$B_1(x^2 + y^2 + z^2) + B_2 x + B_3 y + B_4 z + B_5 = 0 \tag{12}$$

be the equations of any two spheres whose circle of intersection is a generator of the surface S. Therefore A_j and B_j $(j = 1, \cdots, 5)$ are to be regarded as functions of a parameter u. Symbolically, we write $S = [A, B]$. The functions A_j, B_j are five linearly independent solutions of a certain differential equation of the second order, which can, by a proper choice* of the spheres A and B, be written in the canonical form

$$\frac{d^2 A}{du^2} + \frac{d}{du}(lB) + mB = 0. \tag{13}$$

We remark in passing that if S were a surface of type $(0\bar{2}0)$, A_j and B_j would satisfy *two* independent differential equations of the second order, while if S were an annular surface, A_j and B_j would satisfy a differential equation of the *first* order.

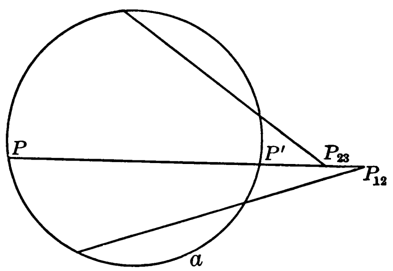

FIG. 3

25. If a_1, $a_2 (= a)$, and a_3 are three consecutive non-singular circles of S, let P_{12} be the center of the sphere σ_{12} that is orthogonal to a_1 and a_2, and let P_{23} be the center of the sphere σ_{23}, orthogonal to a_2 and a_3. Then the lines lying in the plane of a and passing through P_{12} will meet a in a pencil of point-pairs, inverse with respect to σ_{12}; similarly the lines through P_{23} will meet a in another pencil of point-pairs inverse with respect to σ_{23}. These two pencils will have one point-pair in common, namely the pair of points P, P' (see Fig. 3), in which the line $P_{12} P_{23}$ meets a. The net of spheres passing through P and P' is common to the two linear complexes determined by a_1, a_2 and a_2, a_3, respectively. The points P, P' will generate a pair of curves $S^{1,1}$ lying on S, and represented, in F_4, by the planar developable 3-spread $\Sigma^{1,1}$. Hence we can write

$$S^{1,1} = [A, A', B], \tag{14}$$

* See § 1, equation (3).

thus indicating that P, P' are the points of intersection of the three spheres A, A', and B. The sphere passing through a and touching the surface S at the points P, P' is obviously A itself.

26. Now the component curves of $S^{1,1}$ are so related that they have a common tangent circle d at the points P, P'; this circle (see Fig. 4) lies on the sphere A, and either generates an annular surface

$$S^{1,2} = [A, A' + lB], \tag{15}$$

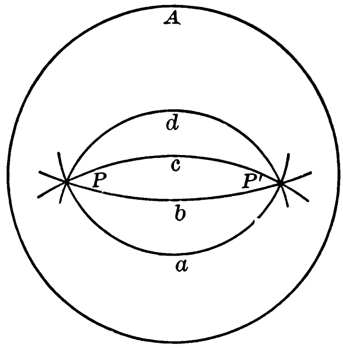

FIG. 4

or is fixed and coincides with $S^{1,1}$. Furthermore, the spheres A either envelope an annular surface $T = [A, A']$, or form a pencil; that is, the characteristic circle b of the sphere A either generates the annular surface T or is fixed and coincides with d; in either case b evidently passes through P and P'.

By translating the theorems of P. D. C., §§ 43 and 44, into the language of circle-geometry, we see that *on every cyclic surface S that is neither annular nor anallagmatic there exists a unique pair of curves $S^{1,1}$, such that at every pair of points in which they meet the same generator of S they have a common tangent circle,* and also that *there exists a unique system of ∞^1 spheres A bitangent tc S such that their points of contact generate $S^{1,1}$.*

We shall call $S^{1,1}$ the pair of *quasi-asymptotic curves* on S. It is clear that neither component of $S^{1,1}$ can degenerate into a fixed point. The variable points P and P' may, however, coincide, so that $S^{1,1}$ becomes a *double* quasi-asymptotic curve; the four circles a, b, c, d are then mutually tangent at the same point. In particular, this will always happen, when every two consecutive circles of S meet in a point P; for then σ_{12} and σ_{23} (§ 25) are point-spheres lying on a. The double quasi-asymptotic curve on S is in that case the locus of the point P.

27. Finally, it is clear that a non-singular circle a and two consecutive circles of S have one and only one *transversal circle c* (cospherical with each of the three), instead of an infinite number, as in the case of an annular or

anallagmatic surface; c will generate a new cyclic surface $\bar{S}$, the transversal of S. From the developments of §§ 2–6 it follows that c lies on the sphere A and passes through the points P, P', that the four circles a, b, c, d form a harmonic pencil (see Fig. 4), and that

$$\bar{S} = [A, A' + \tfrac{1}{2}lB]. \tag{16}$$

The transversal c has 3-point contact with S at each of the points P, P', while b and d have ordinary two-point contact with S at these points.

Hence *two transversal surfaces S and $\bar{S}$ are so related that every circle of one has 3-point contact with the other at each of a pair of points (which may coincide), and that these point-pairs generate the quasi-asymptotic curves $S^{1,1}$ on S.* The harmonic property can be expressed by the

THEOREM. *Every non-singular circle of a cyclic surface S that is neither annular nor anallagmatic and the corresponding circle of its transversal surface $\bar{S}$ are harmonic conjugates with respect to two circles b and d, of which d is the common tangent circle to the two curves of contact of S and $\bar{S}$ at their corresponding points of contact, and b is the characteristic circle of the common bitangent sphere to S and $\bar{S}$ at these points.*

The three principal classes of cyclic surfaces of type $(0\bar{2}10)$ depend on the values of l and m in equation (13), as follows:

Class (a), $l \neq 0$. Class (b), $l = 0$, $m \neq 0$. Class (c), $l = m = 0$.

28. **Class** (a). In this case the four circles a, b, c, d (Fig. 4) are all distinct, and the cyclic surfaces S, T, $\bar{S}$, $S^{1,2}$, generated by them, are therefore distinct. Two consecutive circles d and d' of the annular surface $S^{1,2}$ determine a sphere*

$$C = A' + lB - \frac{m}{l}A, \tag{17}$$

which may be fixed or variable, but which cannot coincide with A, and cannot be tangent to S. Two consecutive circles b and b' of the annular surface T intersect in a pair of points which may or may not be inverse with respect to a fixed sphere, but cannot coincide with P, P', and cannot lie on S. These points generate a pair of curves

$$T^1 = [A, A', (lB)' + mB], \tag{18}$$

associated with S, but not lying on it.

A cyclic surface S of class (a) is evidently characterized by the fact that *every common tangent circle d to its quasi-asymptotic curves at a pair of corresponding points is distinct, in general, from the characteristic circle b of the sphere bitangent to S at those points.*

* See P. D. C., § 45.

Among the circles of the pencil lying on the sphere A and passing through the points P, P' just two generate annular surfaces, namely b and d. The rest generate surfaces of type $(0\bar{2}10)$ and class (a), like S. In particular, *the transversal surface* $\bar{S}$ *is of class* (a), and *its* transversal surface is S itself. All the non-annular surfaces generated by circles lying on the spheres A and passing through the point-pairs P, P' arrange themselves in pairs of transversals.

29. SUBCLASS (a_{11}). This is the most general kind of cyclic surfaces. In this case $S^{1,2}$ is enveloped by ∞^1 spheres C, and is therefore of type $(01\bar{1}110)$; T is also of type $(01\bar{1}110)$; its generators are not orthogonal to a fixed sphere.

SUBCLASS (a_{12}). Again $S^{1,2}$ is of type $(01\bar{1}110)$, while T is now anallagmatic, of type $(01\bar{1}10)$; its generators b and the spheres A are all orthogonal to a fixed sphere σ; the pair of curves T^1 are inverse with respect to σ.

SUBCLASS (a_{21}). Again T is of type $(01\bar{1}110)$, while C (see (17)) is now a fixed sphere, with which $S^{1,2}$ coincides. The circles d and the point-pairs P, P' all lie on the sphere C; so that the quasi-asymptotic curves $S^{1,1}$ are spherical curves.

SUBCLASS (a_{22}). Here $S^{1,2}$ is a sphere, and T is an anallagmatic surface of type $(01\bar{1}10)$.

If the spheres A are planes, S is necessarily of subclass (a_{12}) or (a_{22}); for then T is a developable surface. Its edge of regression (the finite component of T^1) is the locus of the center Q of the sphere which is orthogonal to a and its consecutive circle of S. That is, the pairs of points of contact with S of the spheres passing through a are concurrent with Q.

30. **Class** (b). The circles b and c coincide with d, and the cyclic surfaces T and $\bar{S}$ therefore coincide with $S^{1,2}$. The latter is an annular surface of type $(01\bar{1}110)$, and is therefore neither a sphere nor an anallagmatic surface. The spheres C coincide with the spheres A, and the pair of curves T^1 coincide with the pair of quasi-asymptotic curves $S^{1,1}$.

A cyclic surface S of class (b) is characterized by the fact that *its quasi-asymptotic curves* $S^{1,1}$ *are non-spherical and that every common tangent circle to* $S^{1,1}$ *at a pair of corresponding points is also the characteristic of the sphere bitangent to* S *at those points.*

It is not difficult to show that the quasi-asymptotic curves are in this case ordinary asymptotic curves, while for surfaces of class (a) they are not asymptotic curves.

Since $\bar{S}$ coincides with $S^{1,2}$, *the transversal surface of a cyclic surface* S *of type* $(0\bar{2}10)$ *and class* (b) *is an annular surface of type* $(01\bar{1}110)$, and its generators are the common tangent circles to the quasi-asymptotic curves on S. Conversely, every annular surface $\bar{S}$ of type $(01\bar{1}110)$ has an infinite number of transversal surfaces of class (b). The generators of every one of these lie on the spheres enveloping $\bar{S}$ and pass through the corresponding point-pairs of the curves enveloped by the generators of $\bar{S}$.

31. **Class** (c). The circles b and c again coincide with d; but the latter is now a fixed circle. Hence the quasi-asymptotic curves both coincide with d, and the spheres A form a pencil passing through d.

A cyclic surface of class (c) is characterized by the fact that *its generators are cospherical with a fixed circle d without lying on a fixed sphere and without being orthogonal to a fixed sphere.* Its transversal surface degenerates into the single circle d.

32. **Illustration of class** (a). When a cyclic surface S of class (a) is real, it may happen that one or more of its associated pairs of curves $S^{1,1}$, T^1, or its associated surfaces $S^{1,2}$, T, $\bar{S}$, are imaginary. The fact that all of them may, however, be real will be made clear by the following example.

Consider a helicoidal movement, one of whose path-curves is the helix H whose equations are $x = r\cos\theta$, $y = r\sin\theta$, $z = k\theta$, where

$$k = r\tan\alpha. \tag{19}$$

Also consider a circle a of radius r, whose center is at the origin, whose plane makes an angle $\pi/4$ with the xy-plane, and which has one diameter on the x-axis; a will meet the helix at the point $(r, 0, 0)$. Let S be the surface generated by the circle a as it is carried along by the helicoidal movement.

33. Then it is clear that as θ varies, the plane

$$A : x\sin\theta - y\cos\theta + z - k\theta = 0$$

and the sphere

$$B : x^2 + y^2 + (z - k\theta)^2 = r^2$$

will constantly meet in a generating circle a of S. Writing the homogeneous coördinates (see § 16) of A and B, we have

$$A = (0,\ \sin\theta,\ -\cos\theta,\ 1,\ -k\theta),$$

$$B = (1,\ 0,\ 0,\ -2k\theta,\ k^2\theta^2 - r^2).$$

Differentiating with respect to θ, we find that the pairs of corresponding coördinates of A and B satisfy the differential equation

$$A'' + A + \frac{1}{2k}B' = 0.$$

Although this equation is not quite in the canonical form (13), it is easy to see that the presence of the extra term A will not affect the applicability to this example of the results reached above. Hence $l = 1/2k$, $m = 0$, and S belongs to class (a). In order to find $S^{1,2}$ and $\bar{S}$, we shall need, in view of (15) and (16), to find the spheres $A_1 = A' + B/2k$ and $A_2 = A' + B/4k$.

34. Let us now fix upon a particular generator, $\theta = 0$, of S, and write the corresponding equations of the seven spheres A, B, A', B', A'', A_1, and A_2, referred to rectangular coördinates, as follows:

$$A : y = z, \qquad B : x^2 + y^2 + z^2 = r^2,$$
$$A' : x = k, \qquad B' : z = 0, \qquad A'' : y = 0,$$
$$A_1 : (x + k)^2 + y^2 + z^2 = r^2 + 3k^2,$$
$$A_2 : (x + 2k)^2 + y^2 + z^2 = r^2 + 8k^2.$$

The four spheres B, A', A_2, A_1 intersect the sphere (plane) A in four circles a, b, c, d, respectively, of which b is a straight line. These circles all pass through the pair of points P, P', whose coördinates are

$$[k, \pm \sqrt{\tfrac{1}{2}(r^2 - k^2)}, \pm \sqrt{\tfrac{1}{2}(r^2 - k^2)}].$$

Their centers are on the x-axis and are situated at the distances 0, ∞, $-2k$, $-k$, respectively, from the origin; this verifies the harmonic property. These circles, and therefore the surfaces S, T, $\bar{S}$, $S^{1,2}$, which they generate, are necessarily real. The surface T is a developable, whose edge of regression $T^1 = [A, A', A'']$ is a helix generated by the point $(k, 0, 0)$.

The generator d of $S^{1,2}$ is constantly tangent to the pair of quasi-asymptotic curves $S^{1,1} = [A, A', B]$, generated by P and P'; these are also helices, and are real and distinct, if $k < r$, that is, in view of (19), if the angle α of the helix H is $< \pi/4$. They coincide with each other and with H, if $\alpha = \pi/4$, and become conjugate imaginary curves, if $\alpha > \pi/4$. If $\alpha \mp \pi/4$, the circle c has 3-point contact with S at P and also at P'. Since $S^{1,2}$ is not a sphere, and since T is of type $(01\bar{1}10)$, S and $\bar{S}$ belong to the subclass (a_{12}).

35. **Illustration of class** (b). Out of the material furnished by the preceding example, we can easily construct an example of a cyclic surface of class (b). For since $S^{1,2}$ is an annular surface of type $(01\bar{1}110)$, it is clear that any circle lying on the sphere A_1 and passing through the points P, P', except d, will sweep out, under the influence of the helicoidal movement, a surface of the kind required. In particular, the circle in which the plane $x = k$ meets A_1 will give a simple example.

Pairs of curves in sphere-geometry

36. In the correspondence which we have set up between the projective point-geometry of F_4 and the descriptive sphere-geometry of F_3 a plane of F_4 represents a pair of points of F_3; hence a planar 3-spread represents a pair of curves between which there exists a continuous one-to-one point-correspondence. The two curves may be two parts of the same curve, or, again,

they may be coincident curves, by which is meant that they are generated by pairs of coincident points. On the basis of the study of planar 3-spreads given in P. D. C., §§ 52–59 and in §§ 10–14 of this paper, we shall briefly sketch the descriptive differential sphere-geometry of pairs of curves.

Just as ruled surfaces and planar 3-spreads are dual loci in F_4, so cyclic surfaces and pairs of curves are dual loci in the sphere-geometry of F_3. This duality may be made precise by considering a circle and its foci; as the circle generates a surface, its foci describe a pair of curves,* and vice versa.

Let C^1 and C^2 be any two curves of F_3; and let C denote the two curves taken together, when a continuous point-correspondence has been set up between them. Let P^1 and P^2 be any non-singular pair of corresponding points of C^1 and C^2. We shall call C an annular pair of curves, when C^1 and C^2 have a common tangent circle at every pair of corresponding points P^1, P^2. We proceed to the classification of annular pairs of curves, and then take up the non-annular kind.

37. **Annular pairs of curves.** If the common tangent circle a to C^1 and C^2 is fixed, C^1 and C^2 will simply coincide with parts of the circle a. Apart from this trivial case, C is the envelope of a family of circles, and belongs to one of the four following types.

TYPE $(01\bar{1}0)$. Here C is a pair of spherical and anallagmatic curves. (We shall call two curves *anallagmatic*, if their corresponding points are inverse with respect to a fixed sphere.) Hence C is a pair of curves lying on a fixed sphere and inverse with respect to a fixed circle of the sphere.

TYPE $(01\bar{1}10)$. In this case C is spherical and annular, but not anallagmatic. Hence the enveloping circles cannot be straight lines.

TYPE $(011\bar{1}0)$. Here C is anallagmatic and annular, but not spherical. This is the most general type of anallagmatic curves, and includes the special case in which C has a plane of symmetry, and also the special case in which one of the component curves of C degenerates into a fixed point.

TYPE $(011\bar{1}10)$. Here C is annular, but neither spherical nor anallagmatic. This is the most general type of annular pairs of curves.

NON-ANNULAR PAIRS OF CURVES

38. When C is non-annular, C^1 and C^2 have just one common tangent sphere σ at every pair of corresponding points; σ may be fixed or variable. Thus we have two types.

TYPE $(0\bar{2}0)$. Here S is a pair of curves lying on a fixed sphere, but is neither annular, nor anallagmatic. This is the most general type of spherical curves.

* Cf. Laguerre, *Mémoir sur l'emploi des imaginaires dans la géométrie de l'espace*, Nouvelles annales de mathématiques, ser. 2, vol. 11 (1872), pp. 14–18.

39. TYPE $(01\bar{2}0)$. In this case C is neither annular nor spherical. Two consecutive point-pairs of C are not concyclic and three consecutive point-pairs are not cospherical. Consider three consecutive point-pairs of C, of which the intermediate one is $P = (P^1, P^2)$, and let σ_{12} be the sphere determined by the first and second of these, and σ_{23} the sphere determined by the second and third. Also let a be the circle of intersection of σ_{12} and σ_{23}; a passes through P^1 and P^2, and is cospherical with its consecutive circle; hence it generates an annular surface $C_{1,1}$. *Associated with every pair of curves C of type $(01\bar{2}0)$ is a unique annular surface $C_{1,1}$ whose generating circles pass through the point-pairs of C.*

The sphere determined by the first and third of the above consecutive point-pairs will meet a in a point-pair $R = (R^1, R^2)$, which is the *transversal* of the three point-pairs, in the sense that it is concyclic with each of them; R will describe the transversal pair of curves $\bar{C}$.

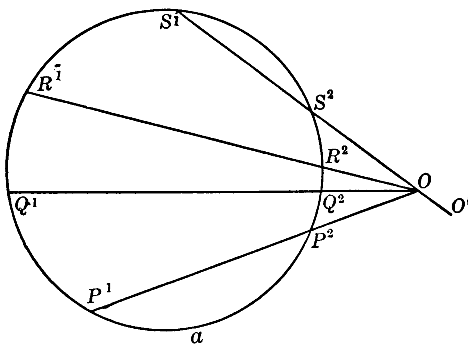

FIG. 5

Let $Q = (Q^1, Q^2)$ be the point-pair in which a meets its consecutive generator of $C_{1,1}$. The three point-pairs P, Q, R (see Fig. 5) belong to a pencil of point-pairs whose vertex O is the center of a sphere with respect to which they are inverse. Its consecutive pencil of point-pairs, whose vertex is O', will have in common with the original pencil just one point-pair $S = (S^1, S^2)$, which is collinear with both O and O'. The four point-pairs P, Q, R, and S form a *harmonic pencil* on the circle a, in the sense that the lines joining them are harmonic.

40. **Class** (a). This is the general case in which the point-pairs P, Q, R, and S are all distinct and describe four pairs of curves $C, C_{1,2}, \bar{C}$, and D, respectively, of which two are annular, $C_{1,2}$ and D. The annular pair $C_{1,2}$ described by (Q^1, Q^2) is either of type $(011\bar{1}10)$ or anallagmatic, of type $(011\bar{1}0)$. On the other hand, the annular pair D described by (S^1, S^2) is either of type $(011\bar{1}10)$ or spherical, of type $(01\bar{1}10)$. The tangent circle to D at the points S^1, S^2 lies on the sphere that envelopes $C_{1,1}$ along the circle a.

The transversal pair of curves $\bar{C}$ is of class (a). Every point-pair lying on a and belonging to the pencil determined by Q and S (except Q and S themselves) will describe a pair of curves of class (a).

41. **Class** (b). The point-pairs R and S coincide with Q, and the annular pair of curves $C_{1,2}$ described by Q is of the general type $(011\bar{1}10)$. Hence $C = S_{1,2}$, that is, the transversal pair of curves is annular. Every point-pair lying on a and belonging to the pencil determined by Q and its consecutive point-pair (except Q itself) will describe a pair of curves of class (b), and having $S_{1,2}$ for its transversal pair of curves.

42. **Class** (c). The point-pairs R and S coincide with Q, and Q is fixed. Hence C may be described as a non-spherical, non-anallagmatic pair of curves, whose point-pairs are all concyclic with a fixed point-pair. The transversal pair of curves degenerates into this fixed point-pair.

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